The Derivative of a Function

In addition to $f'(x)$, which is read as "*f* prime of *x*," other notations are used to denote the derivative of $y = f(x)$. The most common are

$$f'(x), \quad \frac{dy}{dx}, \quad y', \quad \frac{d}{dx}[f(x)], \quad D_x[y].$$

Notations for derivatives

The notation *dy*/*dx* is read as "the derivative of *y with respect to x*" or simply "*dy*, *dx*." Using limit notation, you can write

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x).$$

Example 3 – *Finding the Derivative by the Limit Process*

To find the derivative of $f(x) = x^3 + 2x$, use the definition of the derivative as shown.

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} && \text{Definition of derivative} \\
&= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 + 2(x + \Delta x) - (x^3 + 2x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2x + 2\Delta x - x^3 - 2x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2\Delta x}{\Delta x}
\end{aligned}
$$

Example 3 – *Finding the Derivative by the Limit Process*

cont'd

$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}[3x^2 + 3x\Delta x + (\Delta x)^2 + 2]}{\cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} [3x^2 + 3x\Delta x + (\Delta x)^2 + 2]$$

$$= 3x^2 + 2$$

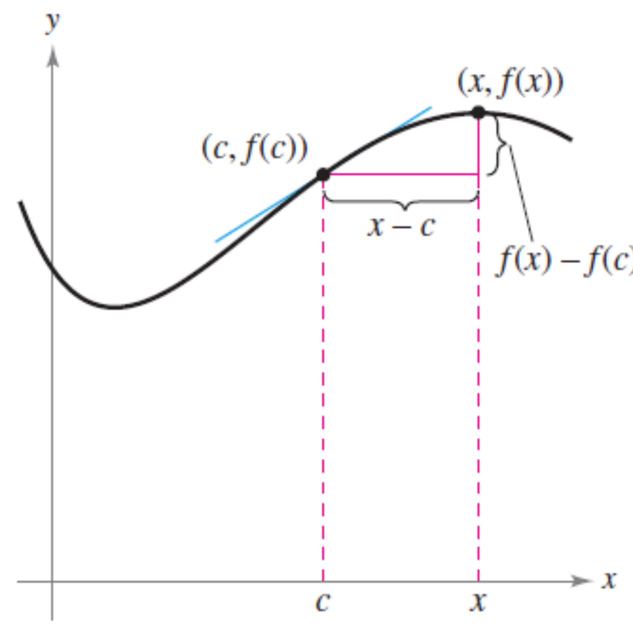
Differentiability and Continuity

Differentiability and Continuity

The alternative limit form of the derivative is useful in investigating the relationship between differentiability and continuity. The derivative of *f* at *c* is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$ Alternative form of derivative

provided this limit exists (see Figure 2.10).



As x approaches c, the secant line approaches the tangent line.

Figure 2.10

Differentiability and Continuity

Note that the existence of the limit in this alternative form requires that the one-sided limits

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \quad \text{and} \quad \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

exist and are equal.

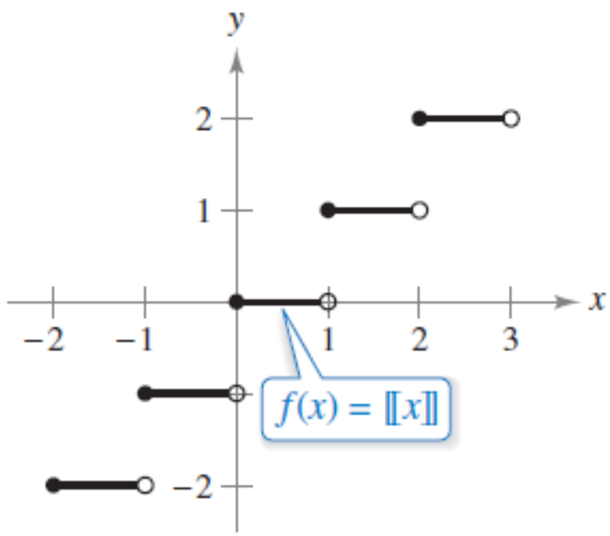
These one-sided limits are called the **derivatives from the left and from the right,** respectively.

It follows that *f* is **differentiable on the closed interval [*a*, *b*]** when if it is differentiable on (*a*, *b*) and when the derivative from the right at *a* and the derivative from the left at *b* both exist.

Differentiability and Continuity

When a function is not continuous at $x = c$, it is also not differentiable at $x = c$.

For instance, the greatest integer function $f(x) = [\![x]\!]$ is not continuous at $x = 0$, and so it is not differentiable at $x = 0$ (see Figure 2.11).



The greatest integer function is not differentiable at $x = 0$ because it is not continuous at $x = 0$.

Figure 2.11

Differentiability and Continuity

You can verify this by observing that

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{[\![x]\!] - 0}{x} = \infty \qquad \text{Derivative from the left}$$

and

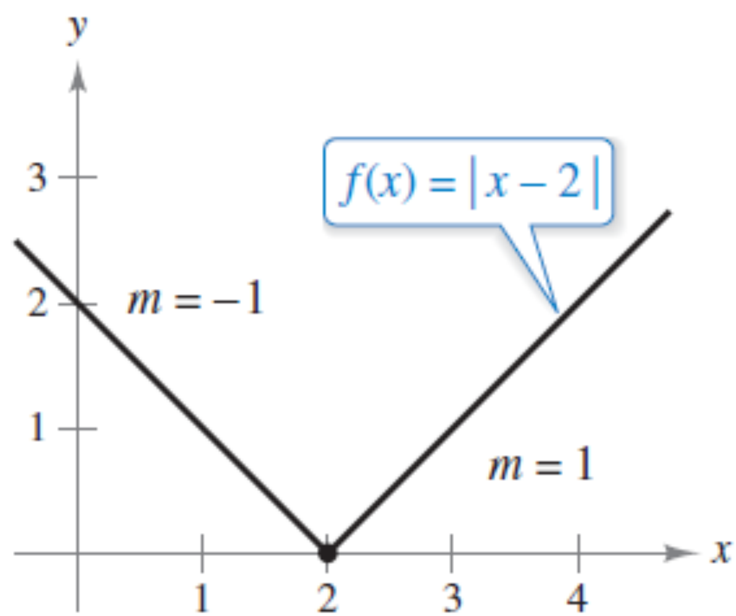
$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{[\![x]\!] - 0}{x} = 0. \qquad \text{Derivative from the right}$$

Although it is true that differentiability implies continuity, the converse is not true.

That is, it is possible for a function to be continuous at $x = c$ and *not* differentiable at $x = c$.

Example 6 – *A Graph with a Sharp Turn*

The function $f(x) = |x - 2|$, shown in Figure 2.12, is continuous at $x = 2$.



f is not differentiable at $x = 2$ because the derivatives from the left and from the right are not equal.

Figure 2.12

Example 6 – *A Graph with a Sharp Turn* cont'd

The one-sided limits, however,

$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{|x - 2| - 0}{x - 2} = -1 \qquad \text{Derivative from the left}$$

and

$$\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{|x - 2| - 0}{x - 2} = 1 \qquad \text{Derivative from the right}$$

are not equal.

So, *f* is not differentiable at *x* = 2 and the graph of *f* does not have a tangent line at the point (2, 0).

Differentiability and Continuity

A function that is not differentiable at a point at which its graph has a sharp turn *or* a vertical tangent line.

THEOREM 2.1 Differentiability Implies Continuity

If f is differentiable at $x = c$, then f is continuous at $x = c$.

The relationship between continuity and differentiability is summarized below.

1. If a function is differentiable at $x = c$, then it is continuous at $x = c$. So, differentiability implies continuity.

2. It is possible for a function to be continuous at $x = c$ and not be differentiable at $x = c$. So, continuity does not imply differentiability.